A Convex Structure on Sofic Embeddings

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Abstract. In [Br] Nathanial Brown introduced a convex-like structure on the set of unitary equivalence classes of unital *-homomorphisms of a separable type II_1 factor into R^ω (ultrapower of the hyperfinite factor). The goal of this paper is to introduce such a structure on the set of sofic representations of groups. We prove that if the commutant of a representation acts ergodicaly on the Loeb measure space then that representation is an extreme point.

Contents

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1 Introduction

Abstract convex-like structures are defined in [Br]. It is shown that $\mathbb{H}om(N, R^\omega)$ posses such a structure, where N is a type II_1 factor and R^ω is the ultrapower of the hyperfinite factor. An important result is that $[\pi] \in \mathbb{H}om(N, R^\omega)$ is an extreme point iff the relative commutant $\pi(N)' \cap R^\omega$ is a factor.

We are interested in a similar convex structure on the set of sofic representations of a group, that we denote by $Sof(G, P^\omega)$, where P stands for permutations. The role of von Neumann algebras will be replaced in this paper by ergodic theory.

We start by briefly introducing the objects that we are working with: sofic groups, Loeb measure space, the space of sofic representations. In the second section we recall what an abstract convex-like structure is and introduce such a structure on $Sof(G, P^\omega)$. Then we prove our main result that ergodicity of the commutant of the sofic representation acting on the Loeb measure space implies that the representation is an extreme point.

1.1 Ultraproducts of matrix algebras

Let ω be a free ultrafilter on $\mathbb{N}$. We shall work with ultraproducts of matrix algebras, which is a particular case of ultraproducts of von Neumann algebras.

Denote by $M_n(\mathbb{C})$ or simply M_n the matrix algebra in dimension n. Recall that for $x \in M_n$ we have the trace norm: $||x||_2 = \left(\frac{1}{n}Tr(x^*x)\right)^{1/2}$ (we normalize it such that $||Id||_2 = 1$ independent of the dimension).

Let $(n_k)_k \subset \mathbb{N}$ be a sequence such that $\lim_{k\to\infty} n_k = \infty$. Define:

$$l^\infty(\mathbb{N}, M_{n_k}) = \{x = (x_k)_k \in \Pi_k M_{n_k} : \sup_k ||x_k|| < \infty\}$$

$$\mathcal{N}_\omega = \{x \in l^\infty(\mathbb{N}, M_{n_k}) : \lim_{k\to\omega} ||x_k||_2 = 0\} \text{ and}$$

$$\Pi_{k\to\omega} M_{n_k} = l^\infty(\mathbb{N}, M_{n_k})/\mathcal{N}_\omega.$$

The ultraproduct $\Pi_{k\to\omega} M_{n_k}$ is a von Neumann algebra, though the proof is a little involved. If $x_k \in M_{n_k}$ we shall denote by $\Pi_{k\to\omega} x_k$ the corresponding element in the ultraproduct. Note that this algebra has a faithful trace, namely $Tr(x) = \lim_{k\to\omega} Tr(x_k)$, where $x = \Pi_{k\to\omega} x_k$.

1.2 The Loeb measure space

The Loeb space was introduced in [Lo] (our exposition is from [El-Sze]). We shall denote by P_n the subgroup of permutation matrices and by D_n the subalgebra of diagonal matrices. We shall interpret $\Pi_{k\to\omega}P_{n_k}$ and $\Pi_{k\to\omega}D_{n_k}$ as subsets in $\Pi_{k\to\omega}M_{n_k}$. By a theorem of Sorin Popa (see [Po], Proposition 4.3) $\Pi_{k\to\omega}D_{n_k}$ is a maximal abelian nonseparable subalgebra of $\Pi_{k\to\omega}M_{n_k}(\mathbb{C})$. It is isomorphic to $L^\infty(X)$ where X is a Loeb measure space. The construction of the Loeb space is valid for any sequence of probability spaces, but we shall need it just for finite spaces.

Let (X_{n_k}, μ_{n_k}) be a space with n_k points equipped with the normalized counting measure such that $(D_{n_k}, Tr) \simeq L^\infty(X_{n_k}, \mu_{n_k})$. Let $p \in \Pi_{k\to\omega}D_{n_k}$ be a projection. It is not difficult to see that $p = \Pi_{k\to\omega}p_k$, where p_k is a projection in D_{n_k} and $Tr(p) = \lim_{k\to\omega} Tr(p_k)$ by definition. A projection in the algebra is the same information as a measurable subset in the underlaying space. This discussion offers a picture of how the Loeb space should be constructed.

Let $(X_{n_k})_\omega$ be the algebraic ultraproduct, i.e. $(X_{n_k})_\omega = \Pi_k X_{n_k}/\sim_\omega$, where $\Pi_k X_{n_k}$ is the Cartesian product and $(x_k) \sim_\omega (y_k)$ iff $\{k : x_k = y_k\} \in \omega$. If $x_k \in X_{n_k}$ we shall denote by $(x_k)_\omega$ the corresponding element in $(X_{n_k})_\omega$. If $A_k \subset X_{n_k}$ then define $(A_k)_\omega = \{(x_k)_\omega : \{k : x_k \in A_k\} \in \omega\} \subset (X_{n_k})_\omega$. Let $\mathcal{B}^0_\omega = \{(A_k)_\omega : A_k \subset X_{n_k}\}$. Then $\mathcal{B}^0_\omega$ is a Boolean algebra of subsets of $(X_{n_k})_\omega$.

For $(A_k)_\omega \in \mathcal{B}^0_\omega$ define $\mu_\omega((A_k)_\omega) = \lim_{k\to\omega} \mu_{n_k}(A_k)$. Let $\mathcal{B}_\omega$ be the completion of $\mathcal{B}^0_\omega$ w.r.t the measure μ_ω. Then $((X_{n_k})_\omega, \mathcal{B}_\omega, \mu_\omega)$ is a nonseparable probability space and $L^\infty((X_{n_k})_\omega, \mu_\omega) \simeq (\Pi_{k\to\omega}D_{n_k}, Tr)$.

1.3 Sofic groups

Introduced by Gromov in [Gr], sofic groups have received considerable attention in the last years. For an introduction to the subject see the nice survey articles of Vladimir Pestov, [Pe],[Pe-Kw].

Definition 1.1. A group G is called *sofic* if there exists a sequence $\{n_k\}_k \subset \mathbb{N}$, $\lim_k n_k = \infty$ and an injective group morphism $\Theta : G \to \Pi_{k\to\omega}P_{n_k}$

The sequence $(n_k)_k$ doesn't have a special role. If such a morphism exists for some $(n_k)_k$ it will exists for any other $(m_k)_k$ as long as $\lim_k m_k = \infty$. The following theorem is due to Gabor Elek and Endre Szabo, [El-Sz1].

Theorem 1.2. *A group G is sofic iff there exists a group morphism $\Theta : G \to \Pi_{k\to\omega}P_{n_k}$ such that $Tr(\Theta(g)) = 0$ for any $g \neq e$.*

Proof. A morphism Θ such that $Tr(\Theta(g)) = 0$ for any $g \neq e$ is clearly injective. For the reverse implication let $\Theta : G \to \Pi_{k\to\omega}P_{n_k}$ be an injective morphism. If $|Tr(\Theta(g)) = 1|$ then $Tr(\Theta(g)) = 1$ and $g = e$. In the end we have $|Tr(\Theta(g))| < 1$ for any $g \neq e$.

Construct $\Theta^{(m)} = \Theta \otimes \Theta \otimes \ldots \otimes \Theta$ (m times tensor product), i.e. $\Theta^{(m)}(g) = \Pi_{k\to\omega}u_g^k \otimes u_g^k \otimes \ldots \otimes u_g^k$, where $\Theta(g) = \Pi_{k\to\omega}u_g^k$. This is a representation of G on $\Pi_{k\to\omega}P_{n_k^m}$. Then $Tr(\Theta^{(m)}(g)) = Tr(\Theta(g))^m$. This means that $Tr(\Theta^{(m)}(g)) \to_{m\to\infty} 0$ for $g \neq e$. A diagonal argument will finish the proof. $\square$

Definition 1.3. A *sofic representation of* G is a group morphism $\Theta : G \to \Pi_{k\to\omega}P_{n_k}$ such that $Tr(\Theta(g)) = 0$ for any $g \neq e$, where $\{n_k\}_k$ is any sequence of natural numbers such that $n_k \to_{k\to\infty} \infty$.

Notation 1.4. Let $\Theta : G \to \Pi_{k\to\omega}P_{n_k}$ be a group morphism, $\Theta = \Pi_{k\to\omega}\theta_k$. Let $\{r_k\}_k$ be a sequence of natural numbers. Define $\Theta \otimes 1_{r_k} : G \to \Pi_{k\to\omega}P_{n_k r_k}$, $\Theta \otimes 1_{r_k} = \Pi_{k\to\omega}\theta_k \otimes 1_{r_k}$. We shall call $\Theta \otimes 1_{r_k}$ an *amplification* of Θ.

Notation 1.5. There is also a *direct sum* of two sofic representations. If $\Theta : G \to \Pi_{k\to\omega}P_{n_k}$, $\Theta = \Pi_{k\to\omega}\theta_k$ and $\Phi : G \to \Pi_{k\to\omega}P_{m_k}$, $\Phi = \Pi_{k\to\omega}\phi_k$ then define $\Theta \oplus \Phi : G \to \Pi_{k\to\omega}P_{n_k+m_k}$ by $\Theta \oplus \Phi = \Pi_{k\to\omega}\theta_k \oplus \phi_k$.

We shall need the following lemma from [Pă]. We also include a short proof for the reader's convenience.

Lemma 1.6. *Let $\{e_i | i \in \mathbb{N}\}$ be projections in $\Pi_{k\to\omega}D_{n_k}$ such that $\sum_i e_i = 1$. Let $\{u_i | i \in \mathbb{N}\}$ be unitary elements in $\Pi_{k\to\omega}P_{n_k}$ such that $v = \sum_i e_i u_i$ is a unitary. Then $v \in \Pi_{k\to\omega}P_{n_k}$.*

Proof. Using the equation $\sum_i e_i = 1$ we can construct projections $e_i^k \in D_{n_k}$ such that:

1. $e_i = \Pi_{k\to\omega}e_i^k$;
2. $\sum_i e_i^k = 1_{n_k}$.

By hypothesis we have $u_i = \Pi_{k\to\omega}u_i^k$ where $u_i^k \in P_{n_k}$. If $v^k = \sum_i e_i^k u_i^k$ then $v = \Pi_{k\to\omega}v^k$, but v^k are not necessary unitary matrices. However v^k is still a matrix only with 0 and 1 entries and exactly one entry of 1 on each row.

We need to estimate the number of columns in v^k having only 0 entries. Denote this number by r_k. Then $v^{k*}v^k$ is a diagonal matrix having r_k entries of 0 on the diagonal. This implies:

$$||v^{k*}v^k - Id||_2^2 \geq \frac{r_k}{n_k}.$$

Because $\Pi_{k\to\omega}v^{k*}v^k = 1$ we have $r_k/n_k \to_{k\to\omega} 0$. We now construct w^k as follows. The matrix v^k has $n_k - r_k$ columns with at least one nonzero entry. For each such column j chose a row i such that $v^k(i,j) = 1$. Let $w^k(i,j) = 1$. In this way we have $n_k - r_k$ nonzero entries in w^k, all of them distributed on different rows and different columns. Choose a bijection between the remaining r_k rows and r_k columns and complete w^k to a permutation matrix. Then:

$$||v^k - w^k||_2^2 = \frac{2r_k}{n_k}.$$

Combined with $r_k/n_k \to_{k\to\omega} 0$ we get $v = \Pi_{k\to\omega}w^k$. This will prove the lemma. $\square$

1.4 The metric space $Sof(G, P^\omega)$

Definition 1.7. For a countable group G define $Sof(G, P^\omega)$ the set of sofic representations of G, factored by the following equivalence relation: $(\Theta_1 : G \to \Pi_{k\to\omega}P_{n_k}) \sim (\Theta_2 : G \to \Pi_{k\to\omega}P_{m_k})$ iff there are two sequences of natural numbers $\{r_k\}_k$ and $\{t_k\}_k$ such that $n_k r_k = m_k t_k$ for any k and there exists $u \in \Pi_{k\to\omega}P_{n_k r_k}$ such that $\Theta_2 \otimes 1_{t_k} = Adu \circ (\Theta_1 \otimes 1_{r_k})$.

Notation 1.8. For a sofic representation Θ we shall denote by $[\Theta]$ its class in $Sof(G, P^\omega)$.

Observation 1.9. By Theorem 2 of Gabor Elek and Endre Szabo from [El-Sz2], the space $Sof(G, P^\omega)$ has exactly one point iff the group G is amenable.

In order to define a metric on $Sof(G, P^\omega)$ we need to fix a counting of the group G. So let $G = \{g_0, g_1, \ldots\}$ where $g_0 = e$. For $[\Theta], [\Phi] \in Sof(G, P^\omega)$ define:

$$d([\Theta],[\Phi]) = inf\{\big(\sum_{i=1}^{\infty}\frac{1}{4^i}||(\Theta\otimes 1)(g_i) - p(\Phi\otimes 1)(g_i)p^*||_2^2\big)^{\frac{1}{2}} : \{n_k\}_k; p \in \Pi_{k\to\omega}P_{n_k}\}$$

The infimum is taken over all the sequences $\{n_k\}_k$ such that the two sofic representations Θ and Φ have amplifications in that dimension. It is clear that this definition does not depend on Θ and Φ, but only of their classes in $Sof(G, P^\omega)$.

Due to a diagonal argument we can see that the infimum in the definition is attain. This implies that d is indeed a distance. Also $Sof(G, P^\omega)$ is complete with this metric.

2 The convex structure on the set of sofic embeddings

Let us first recall what a metric space with a convex-like structure is. The next section is from [Br].

2.1 Metric spaces with a convex-like structure

Let (X, d) be a complete metric space which is bounded (there is a constant C such that $d(x, y) \leq C$ for all $x, y \in X$). In order to provide an abstract convex-like structure on X we need to define the element $t_1x_1 + t_2x_2 + \ldots t_nx_n$, where $x_1, \ldots, x_n \in X$ and $0 \leq t_i \leq 1$ such that $\sum_{i=1}^n t_i = 1$. We shall ask for the following axioms (as Nathanial Brown put it: "properties one would expect if X were an honest convex subset of a bounded ball in some normed linear space")

1. (commutativity) $t_1x_1+\ldots+t_nx_n = t_{\sigma(1)}x_{\sigma(1)}+\ldots+t_{\sigma(n)}x_{\sigma(n)}$ for every permutation $\sigma \in Sym(n)$;
2. (linearity) if $x_1 = x_2$, then $t_1x_1 + t_2x_2 + \ldots + t_nx_n = (t_1 + t_2)x_1 + t_3x_3 + \ldots + t_nx_n$;
3. (scalar identity) if $t_i = 1$, then $t_1x_1 + \ldots + t_nx_n = x_i$;
4. (metric compatibility) $d(t_1x_1 + \ldots + t_nx_n, s_1x_1 + \ldots s_nx_n) \leq C\sum_i |t_i - s_i|$ and $d(t_1x_1 + \ldots + t_nx_n, t_1y_1 + \ldots + t_ny_n) \leq \sum_i t_id(x_i, y_i)$;
5. (algebraic compatibility)
$$t\left(\sum_{i=1}^n t_ix_i\right) + (1-t)\left(\sum_{j=1}^m s_jy_j\right) = \sum_{i=1}^n tt_ix_i + \sum_{j=1}^m (1-t)s_jy_j.$$

In [Ca-Fr], Valerio Capraro and Toblias Fritz proved that this axioms are enough to deduce that X is a closed convex subset in an abstractly constructed Banach space.

2.2 Convex combinations of sofic representations

We now define the convex-like structure on $Sof(G, P^\omega)$.

Definition 2.1. Let $n \in \mathbb{N}$ and for $i = 1, 2, \ldots, n$ let $\Theta_i : G \to \Pi_{k\to\omega}P_{m_k^i}$ be a sofic representation and $\lambda_i \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$. Define $\sum_{i=1}^n \lambda_i\Theta_i$ as follows: choose

natural numbers r_k^i such that $\lim_{k\to\omega} m_k^j r_k^j / \sum_{i=1}^n m_k^i r_k^i = \lambda_j$ for any $j = 1, 2, \dots n$ and set $\sum_{i=1}^n \lambda_i \Theta_i : G \to P_{\sum_{i=1}^n m_k^i r_k^i}$, $\sum_{i=1}^n \lambda_i \Theta_i = \oplus_{i=1}^n (\Theta_i \otimes 1_{r_k^i})$.

Proposition 2.2. *$[\sum_{i=1}^n \lambda_i \Theta_i]$ is well defined, i.e. depends only on $[\Theta_i]$ and λ_i, $i = 1, \dots, n$. We shall denote this object by $\sum_{i=1}^n \lambda_i [\Theta_i]$.*

Proposition 2.3. *The convex structure defined on $Sof(G, P^\omega)$ obeys the axioms $(1) - (5)$ of abstract convex-like structures.*

Proof. Verifications are trivial. Maybe the first part of axiom (4) is a little more technical. □

2.3 Cutting sofic representations

A way of constructing new sofic representations out of old ones is by cutting a representation with a projection from the commutant. This is actually the reverse operation of the direct sum as intruduced in 1.5. Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation, $\Theta = \Pi_{k\to\omega}\theta_k$, where $\theta_k : G \to P_{n_k}$. Let also $p \in \Theta(G)' \cap \Pi_{k\to\omega} D_{n_k}$. Choose projections $p_k \in D_{n_k}$ such that $p = \Pi_{k\to\omega} p_k$. Then $Tr(p_k) = \frac{m_k}{n_k}$, with $m_k \in \mathbb{N}$. Define $\Theta_p : G \to \Pi_{k\to\omega} P_{m_k}$ by $\Theta_p(g) = \Pi_{k\to\omega} p_k \theta_k(g) p_k$. In fact $\Theta_p(g) = p\Theta(g)p = p\Theta(g)$. By definition Θ_p depends on the choice of projections p_k, but it is easy to see that $[\Theta_p]$ does not depend on this choice.

Definition 2.4. If $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ and $p \in \Theta(G)' \cap \Pi_{k\to\omega} D_{n_k}$ then define $\Theta_p : G \to p(\Pi_{k\to\omega} P_{n_k})p$ by $\Theta_p(g) = p\Theta(g)$.

Observation 2.5. If $\Theta = \sum_{i=1}^n \lambda_i \Phi_i$ then there exists $p \in \Theta(G)' \cap \Pi_{k\to\omega} D_{\sum_{i=1}^n m_k^i r_k^i}$ with $Tr(p) = \lambda_i$ such that $[\Phi_i] = [\Theta_p]$.

2.4 Actions on the Loeb space

If $u \in \Pi_{k\to\omega} P_{n_k}$ then $u(\Pi_{k\to\omega} D_{n_k})u^* = \Pi_{k\to\omega} D_{n_k}$. Also the reverse is true: if $x(\Pi_{k\to\omega} D_{n_k})x^* = \Pi_{k\to\omega} D_{n_k}$ then $x = a \cdot u$, where $u \in \Pi_{k\to\omega} P_{n_k}$ and a is a unitary element of $\Pi_{k\to\omega} D_{n_k}$. In an operator language the normalizer of $\Pi_{k\to\omega} D_{n_k}$ is $\mathcal{U}(\Pi_{k\to\omega} D_{n_k}) \cdot \Pi_{k\to\omega} P_{n_k}$.

As $\Pi_{k\to\omega} D_{n_k} \simeq L^\infty((X_{n_k})_\omega, \mu_\omega)$, u defines an automorphism of $((X_{n_k})_\omega, \mu_\omega)$. For a sofic representation we are interested in the action of the commutant on this space. Such actions were considered by David Kerr and Hanfeng Li in [Ke-Li].

Notation 2.6. If $\Theta : G \to \Pi_{k\to\omega}P_{n_k}$ is a sofic representation then we shall denote by $\alpha(\Theta)$ the action of $\Theta(G)' \cap \Pi_{k\to\omega}P_{n_k}$ on the Loeb space $((X_{n_k})_\omega, \mu_\omega)$.

The goal of this article is to link the ergodicity of this action with extreme points in the convex-like structure. We shall now prove that ergodicity is preserved under amplifications. For this we need the next lemma.

Lemma 2.7. *Let $n, r \in \mathbb{N}$ and $A_1, \ldots, A_r \subset \{1, \ldots n\}$. Then exists $A \subset \{1, \ldots, n\}$ and $p \in Sym(r)$ such that:*

$$\sum_{j=1}^{r} |A \triangle A_j| \leq \sum_{j=1}^{r} |A_{p(j)} \triangle A_j|.$$

Proof. For $i = 1, \ldots, n$ let $a_i = |\{j : i \in A_j\}|$. Define $A = \{i : r < 2a_i\}$ (i is an element of A iff more than half sets A_j contain i). Then $\sum_{j=1}^{r} |A \triangle A_j| = \sum_{i=1}^{n} min\{a_i, r - a_i\}$.

For $p \in Sym(r)$ define $R(p) = \sum_{j=1}^{r} |A_{p(j)} \triangle A_j|$. We shall try to evaluate $\sum_{p\in Sym(r)} R(p)$. We want to count how many times $i \in A_{p(j)}$ and $i \notin A_j$.

For $r - a_i$ different values we have $i \notin A_j$; $p(j)$ will be a given fix value for $(r-1)!$ permutations in $Sym(r)$. For another a_i of this values we have $i \in A_{p(j)}$. The number we are looking for is $a_i(r - a_i)(r - 1)!$. It may also happen that $i \notin A_{p(j)}$ and $i \in A_j$. In the end we have:

$$\sum_{p\in Sym(r)} R(p) = \sum_{i=1}^{n} 2a_i(r - a_i)(r - 1)!.$$

It follows that there exists $p \in Sym(r)$ such that $R(p) \geq \sum_{i=1}^{n} 2a_i(r - a_i)/r$. It is easy to see that $min\{a_i, r - a_i\} \leq 2a_i(r - a_i)/r$ so $R(p) \geq \sum_{j=1}^{r} |A \triangle A_j|$. ☐

Proposition 2.8. *Let $\Theta : G \to \Pi_{k\to\omega}P_{n_k}$ be a sofic representation and $\{r_k\}$ a sequence of natural numbers. If $\alpha(\Theta)$ is ergodic then also $\alpha(\Theta \otimes 1_{r_k})$ is ergodic.*

Proof. Let $(S_k)_\omega$ be a subset of the Loeb space $(X_{n_r r_k})_\omega$ such that $u((S_k)_\omega) = (S_k)_\omega$ for any $u \in (\Theta \otimes 1_{r_k})' \cap \Pi_{k\to\omega}P_{n_k r_k}$. Assume that $\mu_\omega((S_k)_\omega) \neq \{0, 1\}$. We shall regard $X_{n_k r_k}$ as r_k copies of X_{n_k}. If u_k is of the form $1_{n_k} \otimes p$, with $p \in Sym(\{1, \ldots, r_k\})$ then u_k will permute this r_k copies of X_{n_k}.

With respect to this partition of $X_{n_k r_k}$ we have $S_k = \sqcup_{j=1}^{r_k} A_k^j$ where $A_k^j \subset X_{n_k}$. Apply the previous lemma to $A_k^1, \ldots, A_k^{r_k}$ to get a set A_k and a permutation p_k. Define $T_k = A_k \times \{1, \ldots, r_k\} \subset X_{n_k r_k}$. Then by the conclusion of the previous lemma we have

$$|T_k \triangle S_k| \leq |(1_{n_k} \otimes p_k)(S_k) \triangle S_k|.$$

Define $u = \Pi_{k\to\omega} 1_{n_k} \otimes p_k$. As $u((S_k)_\omega) = (S_k)_\omega$ by the previous inequality we have $(T_k)_\omega = (S_k)_\omega$.

It is easy to see that $\mu_\omega((T_k)_\omega)$ in $(X_{n_r r_k})_\omega$ is equal to $\mu_\omega((A_k)_\omega)$ in $(X_{n_r})_\omega$. Now the action $\alpha(\Theta)$ is ergodic. So there exists $v \in \Theta' \cap \Pi_{k\to\omega} P_{n_k}$ such that $\mu_\omega(v((A_k)_\omega)\triangle(A_k)_\omega) > 0$. Define $u = v \otimes 1_{r_k}$, $u \in (\Theta \otimes 1_{r_k})' \cap \Pi_{k\to\omega} P_{n_k r_k}$. Again, because we just have an amplification $\mu_\omega(u((T_k)_\omega)\triangle(T_k)_\omega) = \mu_\omega(v((A_k)_\omega)\triangle(A_k)_\omega)$. This contradicts $u((S_k)_\omega) = (S_k)_\omega$ and we are done. $\square$

2.5 Extreme points in the convex structure

We now turn our attention to extreme points in the convex structure. Our first lemma is similar to Proposition 3.3.4 from [Br].

Lemma 2.9. *Let $[\Theta] \in Sof(G, P^\omega)$. Then $[\Theta]$ is an extreme point iff for any projection $p \in \Theta(G)' \cap \Pi_{k\to\omega} D_{n_k}$ $[\Theta] = [\Theta_p]$.*

Proof. If $p \in \Theta(G)' \cap \Pi_{k\to\omega} D_{n_k}$ then $\Theta = \Theta_p \oplus \Theta_{1-p}$. Then, by the definition of the convex structure $[\Theta] = Tr(p)[\Theta_p] + (1 - Tr(p))[\Theta_{1-p}]$. If $[\Theta]$ is an extreme point then $[\Theta] = [\Theta_p]$ for any p.

The reverse implication is a consequence of (2.5): if $\Theta = \lambda_1\Phi_1 + \lambda_2\Phi_2$ then there exists $p \in \Theta(G)' \cap \Pi_{k\to\omega} D_{n_k}$ with $Tr(p) = \lambda_1$ such that $[\Phi_1] = [\Theta_p]$. $\square$

Theorem 2.10. *Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation. Assume that the action $\alpha(\Theta)$ on the Loeb space $((X_{n_k})_\omega, \mu_\omega)$ is ergodic. Then $[\Theta]$ is an extreme point.*

Proof. If $[\Theta]$ is not an extreme point then there exists $\Theta_1, \Theta_2 : G \to \Pi_{k\to\omega} P_{m_k}$ such that $[\Theta_1] \neq [\Theta_2]$ and $\Theta \otimes 1 = \Theta_1 \oplus \Theta_2$. By hypothesis and Proposition 2.8 $\alpha(\Theta \otimes 1)$ is ergodic.

Let $\{p_i\}_i \subset (\Theta_1(G)' \cap \Pi_{k\to\omega} D_{m_k})$ and $\{q_i\}_i \subset (\Theta_2(G)' \cap \Pi_{k\to\omega} D_{m_k})$ be a maximal family of disjoint projections such that $(\Theta_2)_{q_i} = Adu_i \circ (\Theta_1)_{p_i}$ (no tensor, this implies $Tr(p_i) = Tr(q_i)$ as matrix dimensions have to be the same) where u_i is an ultraproduct of permutations. Assume that $\sum_i p_i < 1$ in $\Pi_{k\to\omega} D_{m_k}$.

Let $p = 1 - \sum_i p_i$ and $q = 1 - \sum_i q_i$. Let $\tilde{p}, \tilde{q} \in (\Theta(G) \otimes 1)' \cap \Pi_{k\to\omega} D_{2m_k}$ defined by $\tilde{p} = p \oplus 0$, $\tilde{q} = 0 \oplus q$. Let A_p, A_q be the subsets of $(X_{2m_k})_\omega$ corresponding to the projections $\tilde{p}$ and $\tilde{q}$. Because the action of $(\Theta(G) \otimes 1)' \cap \Pi_{k\to\omega} P_{2m_k}$ is ergodic there exists $u \in (\Theta(G) \otimes 1)' \cap \Pi_{k\to\omega} P_{2m_k}$ such that $\mu_\omega(u(A_p) \cap A_q) > 0$. This is equivalent to $\tilde{q}u\tilde{p} \neq 0$.

Let $v = \tilde{q}u\tilde{p}$, $p_0 = v^*v = \tilde{p}(u^*\tilde{q}u)$ and $q_0 = vv^* = \tilde{q}(u\tilde{p}u^*)$. Then:

$$v(p_0(\Theta \otimes 1)p_0)v^* = vp_0v^*(\Theta \otimes 1) = q_0(\Theta \otimes 1) = q_0(\Theta \otimes 1)q_0.$$

This implies $v((\Theta_1)_{p_0} \oplus 0)v^* = 0 \oplus (\Theta_2)_{q_0}$, so families $\{p_i\}$ and $\{q_i\}$ are not maximal. Then we must have $\sum_i p_i = 1 = \sum_i q_i$. Recall that $(\Theta_2)_{q_i} = Adu_i \circ (\Theta_1)_{p_i}$.

Define $u = \sum_i u_i p_i$. Then $u \in \Pi_{k\to\omega} P_{m_k}$ by 1.6 and $\Theta_2 = Adu \circ \Theta_1$, contradicting $[\Theta_1] \neq [\Theta_2]$. □

Using this theorem (and Theorem 2 from [El-Sz2], see also observation 1.9) we can construct sofic representations such that the commutant acts non-ergodicaly for any sofic non-amenable group.

Question 2.11. *Is a converse of Theorem 2.10 also true?*

2.6 Examples of extreme points

Theorem 2.10 allows us to provide some examples of extreme points. David Kerr and Hanfeng Li proved that $\alpha(\Theta)$ is ergodic for any Θ when G is amenable ([Ke-Li], Theorem 5.8). It follows that any element of $Sof(G, P^\omega)$ is an extreme point. This is possible only if $Sof(G, P^\omega)$ consists of one point, which is consistent with results from [El-Sz2]. The proof of Theorem 5.8 from [Ke-Li] contains something more.

Proposition 2.12. *(Proof of Theorem 5.8,[Ke-Li]) There exists $f : (0,1) \to (0,1)$ a continuous function such that, for any amenable group H, for any sofic representation $\Theta : H \to \Pi_{k\to\omega} P_{n_k}$ and any $Y = (Y_k)_\omega$, $Z = (Z_k)_\omega$ subsets of $(X_{n_k})_\omega$ of strictly positive measure there exists $u \in \Theta'(H) \cap \Pi_{k\to\omega} P_{n_k}$ such that:*

$$\mu_\omega(u(Y) \cap Z) \geq f(min(\mu_\omega(Y), \mu_\omega(Z))).$$

Using this proposition we can prove the existence of extreme points for initially sub-amenable groups.

Theorem 2.13. *Let G be an initially sub-amenable group. Then there exists Θ a sofic representation of G such that $\alpha(\Theta)$ is ergodic.*

Proof. For this proof we need product ultrafilter techniques (see [Ca-Pă]). The drawback of the proof is that we start with an ultrafilter ω and get in the end an extreme point for $\omega \otimes \omega$, in other words an extreme point in the set $Sof(G, P^{\omega\otimes\omega})$.

Let $G = \cup_k F_k$, where $\{F_k\}_k$ is an increasing sequence of finite subsets of G. By hypothesis for each k there exists H_k an amenable group and $\phi_k : F_k \to H_k$ such that $\phi(g)\phi(h) = \phi(gh)$ when $g, h, gh \in F_k$. Choose $\Theta_k : H_k \to \Pi_{r\to\omega} P_{n_k,r}$ a sofic representation of H_k, $n_{k,r} \in \mathbb{N}$.

The ultraproduct $\Pi_{r\to\omega} P_{n_k,r}$ is a metric group for each k. We can construct the ultraproduct of these metric groups to get the equality:

$$\Pi_{k\to\omega}(\Pi_{r\to\omega} P_{n_k,r}) = \Pi_{(k,r)\to\omega\otimes\omega} P_{n_k,r}.$$

Define $\Theta : G \to \Pi_{(k,r)\to\omega\otimes\omega} P_{n_k,r}$ by:

$$\Theta(g) = \Pi_{k\to\omega}\Theta_k(\phi_k(g)).$$

We now prove that $\alpha(\Theta)$ is ergodic. Let $Y = (Y_{k,r})_{\omega\otimes\omega}$, $Z = (Z_{k,r})_{\omega\otimes\omega}$ be subsets of $(X_{n_{k,r}})_{\omega\otimes\omega}$ of strictly positive measure. Define $Y_k = ((Y_{k,r})_r)_\omega$ and $Z_k = ((Z_{k,r})_r)_\omega$. Then $\mu_{\omega\otimes\omega}(Y) = \lim_{k\to\omega}\mu_\omega(Y_k)$ and the same for Z. Because H_k is amenable, by Proposition 2.12, there exists $u_k \in \Theta_k'(H_k) \cap \Pi_{r\to\omega} P_{n_{k,r}}$ such that:

$$\mu_\omega(u_k(Y_k) \cap Z_k) \geq f(min(\mu_\omega(Y_k), \mu_\omega(Z_k))).$$

Let $u = \Pi_{k\to\omega} u_k$. Then u commutes with Θ and by continuity:

$$\mu_{\omega\otimes\omega}(u(Y) \cap Z) \geq f(min(\mu_{\omega\otimes\omega}(Y), \mu_{\omega\otimes\omega}(Z))) > 0.$$

$\square$

In [Cor], Yves Cornulier constructed a sofic group that is not initially sub-amenable, so the last theorem doesn't solve the problem of existence of extreme points for sofic groups in general.

Residually finite groups are initially sub-amenable, but in this case there is an easier way of constructing extreme points.

Theorem 2.14. *(Theorem 5.7,[Ke-Li]) Let G be a residually finite group and let $\{G_i\}_{i\in\mathbb{N}}$ be a sequence of finite index normal subgroups such that $\cap_{n\in\mathbb{N}}\cup_{i\geq n} G_i = \{e\}$. Let Θ be the sofic representation of G constructed by taking the left action of G on G/G_i. Then $\alpha(\Theta)$ is ergodic.*

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